Mathematical Foundations of Arc Length-Based Aspect Ratio Selection

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Abstract

The aspect ratio of a plot can strongly influence the perception of trends in the data. *Arc length based aspect ratio selection* (AL) has demonstrated many empirical advantages over previous methods. However, it is still not clear why and when this method works. In this paper, we attempt to unravel its mystery by exploring its mathematical foundation. First, we explain the rationale why this method is parameterization invariant and follow the same rationale to extend previous methods which are not parameterization invariant. As such, we propose *maximizing weighted local curvature* (MLC), a parameterization invariant form of *local orientation resolution* (LOR) and reveal the theoretical connection between *average slope* (AS) and *resultant vector* (RV). Furthermore, we establish a mathematical connection between AL and banking to 45 degrees and derive the upper and lower bounds of its average absolute slopes. Finally, we conduct a quantitative comparison that revises the understanding of aspect ratio selection methods in three aspects: (1) showing that AL, AWO and RV always perform very similarly while MS is not; (2) demonstrating the advantages in the robustness of RV over AL; (3) providing a counterexample where all previous methods produce poor results while MLC works well.

Index Terms: I.3.7 [Computing Methodologies]: Computer Graphics—Three-Dimensional Graphics and Realism

1 Introduction

The aspect ratio of a plot (height/width) can heavily influence the visual perception of trends in given data. Taking line charts as an example, changes in aspect ratio induce changes in the orientations of line segments, which strongly affects the ability of the viewer in judging rates of change [12]. Hence, choosing an appropriate aspect ratio is essential to reveal important patterns in the data. William Cleveland [4] pioneered the principle of *banking to 45°*, which laid the perceptual foundation for aspect ratio selection. It assumes that centering the orientations of the plots line segments around 45 degrees can minimize the error in visual judgement of slope ratios. Based on this principle, Cleveland et al. [1, 2, 3] proposed two methods: *median slope* (MS) and *weighted average orientation* (AWO), where AWO generally yields reasonable aspect ratios. Following the formulation of MS, Heer and Agrawala [6] suggested to set the *average slope* (AS) to be one. To further improve the visual perception, they proposed to maximize the angle between line segments, including two methods: *global orientation resolution* (GOR) and *local orientation resolution* (LOR). All these methods choose the aspect ratio based on adjusting the orientations or slopes of line segments.

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Table 1: Comparison between various aspect ratio selection methods, where AO and AWO, AS and RV, LOR and MLC are three paired parameterization variant and invariant forms, "?" represents that we differ in our observation to Talbot et al. [16].

	Parameterization Invariant	Symmetry Preservation	Robust
AO	×	×	×
AWO	✓	×(?)	×
AS	×	×	×
RV	✓	✓	✓
LOR	×	×	×
MLC	✓	×	×
GOR	×	×	×
MS	×	×	×
AL	✓	✓	✓(?)

Motivated from the geometric observation, Guha and Cleveland [13] suggested the *resultant vector* (RV) that uses the ratio of the total variation of line segments in the y and x directions as the aspect ratio. Likewise, Talbot et al. [15] proposed an alternative method, *arc length based aspect ratio selection* (AL), which determines the aspect ratio by minimizing the arc length while keeping the area of the plot constant. As shown in Table 1, this method has demonstrated many empirical advantages over previous methods, such as parameterization invariance, symmetry preservation and robustness. Among them, the property of parameterization invariance is quite important, since it indicates the invariance of the selected aspect ratio under changes to the parameterization of the curve. However, many aspects of this method are still unclear. For example, it is not clear why this method is parameterization invariant or if there is a connection between this method and the principle of banking to 45°.

In this paper, we attempt to unravel the mystery of AL by delving into its mathematical foundation. We first unveil its property of parameterization invariance with a line integral representation. By extending LOR and AS to be parameterization invariant with such representation, we present a new aspect ratio algorithm, *maximizing weighted local curvature* (MLC), and reveal a theoretical connection between an extension of AS and RV. We establish the connection between AL and the principle of banking to 45° and derive the upper and lower bounds of its produced averaged absolute slopes. Finally, we conduct a comprehensive quantitative comparison between the five methods AL, AWO, RV, MS, MLC, with similar experimental setting of Talbot et al. [15]. Our results contribute to the understanding of aspect ratio selection methods in three aspects: first, we show that AWO always performs similarly to AL, while MS is not. Secondly, we find that RV is very similar to AL but faster and more robust in handling zero-length segments. Lastly, we provide a counterexample where all previously proposed aspect ratio methods produce poor results while MLC works well.

2 Related Work

Cleveland et al. [4] studied aspect ratio selection systematically for the first time. They conducted human-subject experiments and observed that the visual judgement of slope ratios between adjacent line segments is most accurate when the *orientation resolution* between segments is maximized. They also found that the orientation resolution is maximized when the average orientation between them is 45°. Based on these two observations, Cleveland et al. [1, 2, 3] and the following Heer and Agrawala [6] as well as Talbot et al. [15] proposed a few methods for aspect ratio selection with different properties, see Table 1.

For comparing these methods, we represent the input curve by a sequence of adjacent line segments $\{(\Delta x_1, \Delta y_1), \cdots, (\Delta x_n, \Delta y_n)\}$, where Δx_i and Δy_i are the lengths of the i-th line segment in x and y directions.

Median absolute slope (MS) [4] chooses the aspect ratio such that the median absolute slope of the line segments is one. Likewise, Heer and Agrawala [6] suggest to set the aspect ratio as the reciprocal of the *average absolute slope* (AS). Since our perceptual processes are more sensitive to orientation than slope, Cleveland et al. [1, 2, 3] suggest choosing the aspect ratio such that the *average absolute orientation* (AO) approximates 45°. However, this method generates different results for the same curve with different parameterizations. To avoid this problem, they later propose the *weighted average absolute orientation* (AWO), that weights the orientation of each line segment with the line segment length:

$$\frac{\sum_i |\theta_i(\alpha)| l_i(\alpha)}{\sum_i l_i(\alpha)} = 45^\circ \tag{1}$$

where θ_i and l_i are the orientation and length of the i-th line segment, respectively. Cleveland et al. [1, 2, 3] concluded that AWO can generate a more satisfactory aspect ratio than other methods.

Rather than banking the line orientation to 45°, Heer and Agrawala [6] propose to directly maximize the *orientation resolution* between line segments, which is defined as the smallest angle between two line segments:

$$\gamma_{i,j} = \min(|\theta_i(\alpha) - \theta_j(\alpha)|, 180 - |\theta_i(\alpha) - \theta_j(\alpha)|). \tag{2}$$

A method that maximizes $\gamma_{i,j}$ between all paired line segments is called *global orientation resolution* (GOR),

$$\max \sum_i \sum_j \gamma_{i,j}^2, \tag{3}$$

which produces in most practical cases similar results to AO but is quite expensive. In contrast, *local orientation resolution* (LOR) only maximizes $\gamma_{i,j}$ between successive line segments

$$\max \sum_{i=1}^{n-1} \gamma_{i,i+1}^2, \tag{4}$$

which is more efficient. However, both methods have two drawbacks. First, they cannot handle perfectly horizontal and vertical segments so that such segments must be culled first. Second, they are not invariant to parameterization changes where different samplings of the input curves will lead to different aspect ratios. In this paper, we extend LOR to be parameterization invariant and show its relationship with local curvature.

Guha and Cleveland [13] proposed the *resultant vector* (RV) method, which banks the line to 45° with a simple tractable algebraic form:

$$a = \frac{\sum_i |\Delta x_i|}{\sum_i |\Delta y_i|}. \tag{5}$$

Geometrically, this method first reflects each line segment to a vector in the first quadrant and then selects the aspect ratio to set the slope of the resultant vector to be one. In this paper, we provide an alternative interpretation of this method by showing that it can also be derived by generalizing AS to a continuous representation.

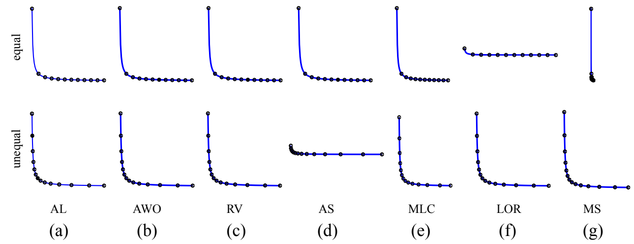

Figure 1: Comparison of the aspect ratio methods on the curve $y = 1/x$, where the line segments in the above row are equally spaced on x axis while the ones in the bottom row are non-equally spaced. AL, AWO and RV are not only parameterization invariant but also symmetry preservation. MLC generates similar but different aspect ratios for two parameterizations due to a local optimal solution, although it is essentially parameterization invariant.

Moreover, we found it performs similarly to AL but is faster and more robust.

Rather than following the 45° principle, the *arc length based method* (AL) [15] chooses the aspect ratio by minimizing the arc length of the plotted curve while keeping the plotted area as a constant:

$$\min_{a \in (0,\infty)} \sum_{i=1}^{n} || \frac{\Delta x_i}{\sqrt{a}}, \sqrt{a} \Delta y_i ||. \tag{6}$$

AL has demonstrated many empirical advantages over previous methods, such as parameterization invariance, symmetry preservation, and robustness. Talbot et al. [15] found that among all existing aspect ratio selection methods, only AL can preserve symmetry for a curve that is symmetric around $y = x$. They generalize the aspect ratio methods for 2D contour plots and show that AL performs similarly to MS by banking circles to circles, whereas AWO produces ellipses. Unfortunately, the mathematical and perceptual principles behind this method are still unknown. We explore its mathematical foundation, and conduct an experiment to compare AL with another methods.

Talbot et al. [16] expand the experimental design of Cleveland et al. [4] and show that banking to 45° is not necessarily the best choice. Inspired by this result, Fink et al. [5] recently proposed to select the aspect ratio for 2D scatter plots based on Delaunay triangulation. They argue that a proper aspect ratio should result in a Delaunay triangulation with aesthetic geometric properties. Our work can serve as a complement to them, since we attempt to find the mathematical foundations of AL.

3 Line Integral: Parameterization Independence

In this section, we introduce a line integral representation in order to explain why these methods are parametrization invariant and extend LOR and AS using this representation to achieve invariance.

3.1 Line Integral Representation

Given a scalar function $f : D \subset R^n \to R$, the line integral [17] along a curve $C \subset D$ is defined as:

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt, \tag{7}$$

where C is parameterized by the function $\mathbf{r}(t)$ $a \leq t \leq b$ whose derivative is $\mathbf{r}'(t)$. As long as the function f is unchanged, the value of this integral is the same for a specific function f, no matter how the curve C is parameterized by the function $\mathbf{r}$. In other words, the integral of the scalar function over a curve C is independent of the parametrization $\mathbf{r}$ of C. If we are able to formulate the aspect ratio

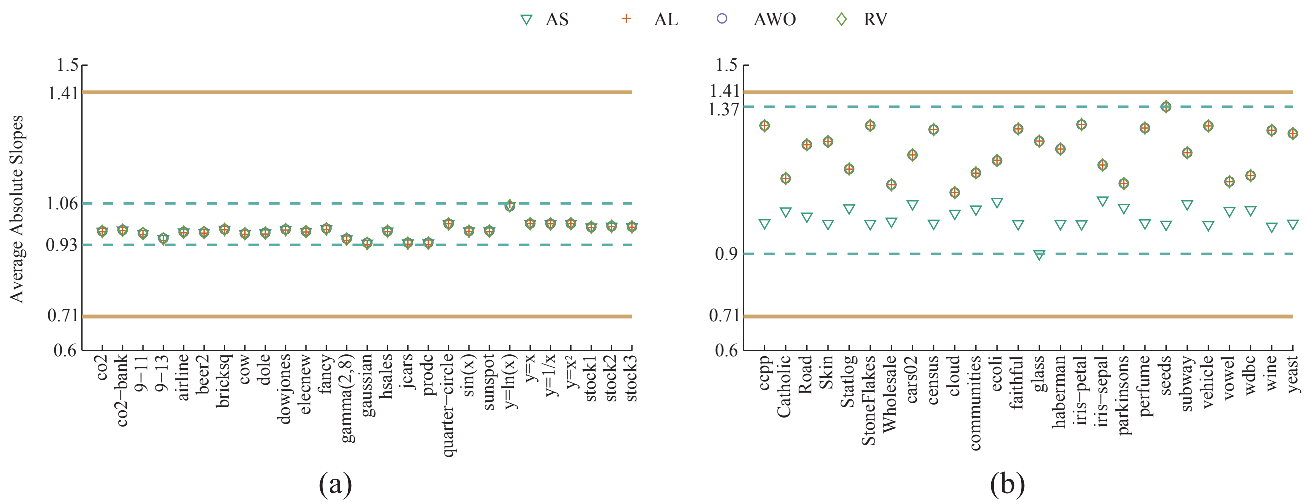

Figure 2: The average absolute slopes generated by applying various aspect ratio selection methods to 1D uniformly sampled curves (a) and 2D non-uniformly sampled contours (b). AL, AWO and RV are similar for 1D and 2D data, while AS is similar with AL for 1D curves but performs differently for 2D contours.

selection problem with a line integral representation, the selected aspect ratio will be invariant to parameterization changes.

Since the arc length is inherently a variable of the line integral, Equation 6 can be represented as follows:

$$\begin{aligned}\min_{a\in(0,\infty)} \sum_{i=1}^{n} \frac{1}{\sqrt{a}} ||\Delta x_i, a\Delta y_i|| \\ = \frac{1}{\sqrt{a}} \int_C ds. \end{aligned} \quad (8)$$

Here, the function f has a constant value (one for simplicity) in the numerator. This integral representation provides an alternative, intuitive interpretation of AL: finding the largest squared root of the aspect ratio that produces the shortest arc length. This indicates that the area-preserved constraint of AL is not necessary and thus we do not have to simultaneously scale the x and y axes.

Similarly, AWO can be written as the following minimization:

$$\begin{aligned}\min_{a\in(0,\infty)} \left| \frac{\sum_i |\theta_i(a)| l_i(\alpha)}{\sum_i l_i(a)} - 45^\circ \right| \\ = \left| \frac{\int_C |\theta(\alpha)| ds}{\int_C ds} - 45^\circ \right| \end{aligned} \quad (9)$$

where the integrands in the numerator and denominator are $|\theta(a(s))|$ and one, respectively. Here, the line orientation is represented as a function of the arc length. With that, we can show that AL and AWO both can be represented as line integrals, which are independent of parameterization as demonstrated in Figure 1(a,b).

3.2 Maximizing Weighted Local Curvature

The curvature of a curve C can be estimated by the change of slope angles of the tangent line at a given point [11]. Since the angle change is often described as the smallest acute angle between two successive line segments along the arc length, the curvature κ at p_i can be defined as:

$$\kappa_i = ||\frac{r_{i,i+1}}{ds}||$$

where $r_{i,i+1}$ is the orientation resolution between the i-th and $i+1$-th line segments, ds is the distance between these point p_i and p_j. Hence, the curvature-based line integral can be expressed as follows:

$$\int_C |\frac{d\theta}{ds}| ds = \int_C \frac{|\theta_i - \theta_{i+1}|}{ds} ds = \sum_i |\theta_i - \theta_{i+1}|. \quad (10)$$

We call this method *maximizing weighted local curvature* (MLC). Equations 4 and 10 show that they both sum up $r_{i,i+1}$ but MLC uses the L1 norm while LOR uses the L2 norm. Moreover, the L1 norm allows us to represent the maximal orientation resolution as a curvature-based line integral that is parameterization invariant. In practice, we found that MLC is not able to find the same aspect ratio for the same data under different parameterizations (see Figure 1(e)), since Equation 10 easily traps in local optima.

3.3 The relationship between AS and RV

AS banks the average absolute slope to 1 using the following Equation:

$$a = \frac{n}{\sum_i |m_i|} \quad (11)$$

where m_i is the slope of the i-th line segment. Regarding the slope m as a function of x on the closed interval $[x_1, x_2]$, we aim at generalizing AS to determine an optimal aspect ratio for continuous input data.

Assuming the data points are equally spaced along x direction with a step Δx, Equation 11 becomes:

$$a = \frac{n\Delta x}{\sum_i |m(x_i)| \Delta x} = \frac{x_2 - x_1}{\sum_i |m(x_i)| \Delta x}$$

where $\Delta x = (x_2 - x_1)/n$. As n goes to infinity, we yield

$$a = \lim_{n\to\infty} \frac{x_2 - x_1}{\sum_i^n |m(x_i)| \Delta x} = \frac{x_2 - x_1}{\int_{x_1}^{x_2} |m(x)| dx} = \frac{\int_{x_1}^{x_2} dx}{\int_{x_1}^{x_2} |m(x)| dx}. \quad (12)$$

According to the Pythagoras' theorem, we have $(ds)^2 = (dx)^2 + (dy)^2$ and can derive that $dx = \cos(\theta(s))ds$, which enables us to rewrite Equation 12 as

$$a = \frac{\int_{c_1}^{c_2} |\cos(\theta(s))| ds}{\int_{c_1}^{c_2} |\sin(\theta(s))| ds} \tag{13}$$

$$= \frac{\sum_i |\Delta x_i|}{\sum_i |\Delta y_i|}. \tag{14}$$

Comparing with Equation 5, we see that Equation 14 has the same formulation as the RV method [13], but is derived from different background. The resultant vector is geometrically motivated whereas Equation 13 is obtained by extending AS using line integral representation. This extension explains why RV is parameterization invariant. Since AS is derived from the 45° principle, RV also banks the slope to one and thus it preserves the symmetry like AL. Figure 1(c) demonstrates these advantages using a curve $y = 1/x$.

4 Connection Between AL and Banking to 45°

Although AL produces results similar to AWO for time series like curves, Talbot et al. [15] have not given any perceptual reasons for using AL. In this section we show that AL has a tendency to satisfy the wanted property of banking to 45 degree by establishing the mathematical connection between them.

4.1 Connection to 45° Principle

Considering a single line segment, Equation 6 becomes:

$$\min_{a\in(0,\infty)} \sqrt{(\Delta x_i/\sqrt{a})^2 + (\sqrt{a}\Delta y_i)^2}$$
$$\geq \sqrt{2}\Delta x_i \Delta y_i, \tag{15}$$

where the equality holds when $a = \Delta x_i/\Delta y_i$ and the resulting line orientation is 45°. This indicates that AL banks the line segment to 45° by enforcing that the triangle corresponding to each line segment is an isosceles right triangle (see Figure 3). Generalizing Equation 15 to multiple line segments, different values for Δx_i and Δy_i typically result in a different optimal a, so we cannot individually optimize each line segment. To optimize over all line segments, AL attempts to choose a in a way that summing Equation 15 over n line segments reaches a minimum. This method has a similar spirit to AWO, which chooses a in a way that the weighted mean is 45°.

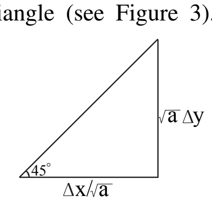

Figure 3: An isosceles right triangle.

For equally spaced data points, Δx_i has a constant value. In this case, Equation 6 can be written as:

$$\min_{a\in(0,\infty)} \sum_{i=1}^{n} || \frac{\Delta x_i}{\sqrt{a}}, \sqrt{a}\Delta y_i ||$$
$$= \frac{\sum_{i=1}^{n} \sqrt{1 + a^2|m_i|^2}}{\sqrt{a}} \Delta x_i, \tag{16}$$

where $|m_i|$ is the absolute slope of the i-th line segment. Since large $|m_i|$ would create a large arc length, this optimization banks the line segments with larger absolute slopes to 45° in a more aggressive way.

4.2 The Bounds of AL's Average Absolute Slope

As discussed above, AL and AWO both attempt to select the aspect ratio such that most of the line segments are banked to 45°. However, it is not obvious whether the aspect ratio produced by Equation 16 really converges to 1. Now we show that AL has a tendency to satisfy banking to 45° by computing the bounds of the average absolute slopes generated by AL.

Assuming uniformly sampled data points, the derivative of Equation 16 with respect to a to zero yields:

$$\sum_{i=1}^{n} \frac{a^2 m_i^2 - 1}{\sqrt{a^2 m_i^2 + 1}} = 0, \tag{17}$$

which can be rewritten as

$$\sum_{i=1}^{n} \frac{(a|m_i| - 1)(a|m_i| + 1)}{\sqrt{(a|m_i|)^2 + 1}} = 0. \tag{18}$$

Since $a|m_i|$ is always non-negative, $a|m_i| + 1$ and $\sqrt{(a|m_i|)^2 + 1}$ must be larger than 1. Putting them together as ω_i, we can derive their bounds:

$$1 \leq \omega_i = \frac{a|m_i| + 1}{\sqrt{(a|m_i|)^2 + 1}} = \sqrt{1 + \frac{2a|m_i|}{a^2 m_i^2 + 1}} \leq \sqrt{1 + \frac{2a|m_i|}{2a|m_i|}} = \sqrt{2}. \tag{19}$$

We see that ω_i reaches the minimum of 1 when $a|m_i|$ is zero. On the other side, ω_i reaches a maximum of $\sqrt{2}$ when $a|m_i|$ is one. Substituting ω_i into Equation 18, we get

$$\sum_{i=1}^{n} (a|m_i| - 1)\omega_i = 0, \tag{20}$$

where a is undefined if all $|m_i|$ are zeroes. If not all $|m_i|$ are zeroes, a is defined as

$$a = \frac{\sum_i \omega_i}{\sum_i \omega_i |m_i|}. \tag{21}$$

Multiplying both sides of Equation 21 by $\sum_i |m_i|/n$ we get

$$\frac{a \sum_i |m_i|}{n} = \frac{\sum_i \omega_i \sum_i |m_i|}{n \sum_i \omega_i |m_i|}, \tag{22}$$

where the left side is the average absolute slope.

Since ω_i is a value between 1 and $\sqrt{2}$, the minimal value of Equation 22 can be obtained by setting all ω_i in the numerator and denominator to 1 and $\sqrt{2}$, respectively. Likewise, the maximal value can be obtained by setting all ω_i in the numerator and denominator to $\sqrt{2}$ and 1, respectively. Thus, the interval of the average absolute slope is $(\sqrt{2}/2, \sqrt{2})$, which corresponds to $(35.3°, 56.7°)$. Since the value of ω_i cannot be different in the numerator and denominator, the actual range is smaller.

Although the derivation of these bounds is based on uniformly sampled data points, it is also applicable to non-uniformly sampled data, because AL is independent of the parameterization. Figure 2 shows the average absolute slopes generated by applying various methods to uniformly sampled 1D curves (Figure 2(a)) and non-uniformly sampled 2D contours (Figure 2(b)). Since the average absolute slopes generated by MS and MLC exceed the range, we did not shown them. The average absolute slopes generated by all methods are within the bounds we derived, the range generated with 2D contours is somewhat larger. From Figure 2(b), we find that the aspect ratios selected by AL, AWO and RV generate almost the same average absolute slopes while AS produces different results for non-uniformly sampled 2D contours because AS is not parametrization invariant.

5 QUANTITATIVE EVALUATION

In this section, we mainly answer the following questions:

- Does AWO behave differently from AL for contour plots?
- Does MS always perform similarly to AL for contour plots?
- Is there any method that has similar performance with AL but is faster and more robust?
- Is there any counterexample where all previously methods produce poor results?

To answer these questions, we implemented AL, AWO, AS, RV, MS and MLC in C++. The optimizations involved in AL, AWO and MLC are solved by the method-of-moving-asymptotes (MMA), see [14] which is provided by the NLopt library [8] and converges to a global optimum. Like Talbot et al. [15], we parameterize the optimization search of AL with $\log(a)$, which converges faster than directly searching for a.

We performed a comprehensive quantitative comparison between AL and the other five methods. Besides the data sets used by Heer and Agrawala [6] and Talbot et al. [15], we downloaded a few stock value data sets and 2D data sets from the UCI dataset [10]. In total we tested 27 1D curves and 26 2D contours, where the 2D contour lines are generated with a grid-based kernel density estimator [9] used by Talbot et al. [15]. All codes and tested data sets are included in the supplemental material.

5.1 Overall Comparison

Figure 4(a) shows the negative logarithm of the relative aspect ratios chosen by these methods. A selection of 1D curves and 2D contours is shown in Figures 5 and 6. To quantitatively compare different methods, we compute the averaged relative errors between AL and other methods by using

$$\text{error} = \log\Big(\frac{1}{N}\sum_{i}^{N}\frac{X_i - AL_i}{AL_i}\Big), \tag{23}$$

where X refers to different methods, X_i is the aspect ratio selected by method X for the i-th curve, and N is the number of tested data sets.

Figure 4(b) displays a scatterplot showing relative errors computed from 1D curves and 2D contours. We see that AL and AWO always perform very similar for 1D curves or 2D contours, while MS is not quite similar with AL for some contours.

This observation is quite different from Talbot et al. [15], since they found that AL behaves differently from AWO for 2D contours but essentially is the same as MS. One of their arguments is that AL and MS both bank a circle to a circle while AWO produces an ellipse. In the following, we deeply explore this inconsistent observation.

5.2 AL vs. AWO

In order to verify whether AWO banks a circle to an ellipse, we generate a few ellipses with varying ratios of major to minor axis in the range of 0.1-2 and then explore how this ratio influences the selected aspect ratios. Figure 7 demonstrates that AWO and RV both generate almost the same aspect ratios as AL and the selected aspect ratios are roughly the same as the input ratios. This implies that AWO, RV and AL all bank ellipses to circles, let alone banking circles to circles. As discussed in Section 4, AWO and AL both bank line segments to 45°. As for curves with symmetric shapes, symmetry preserved banking is the most parsimonious action. Thus, we think these two methods both allow to preserve important symmetric shapes. For asymmetric shapes (Figure 5 and Figure 6), these two methods select almost the same aspect ratios.

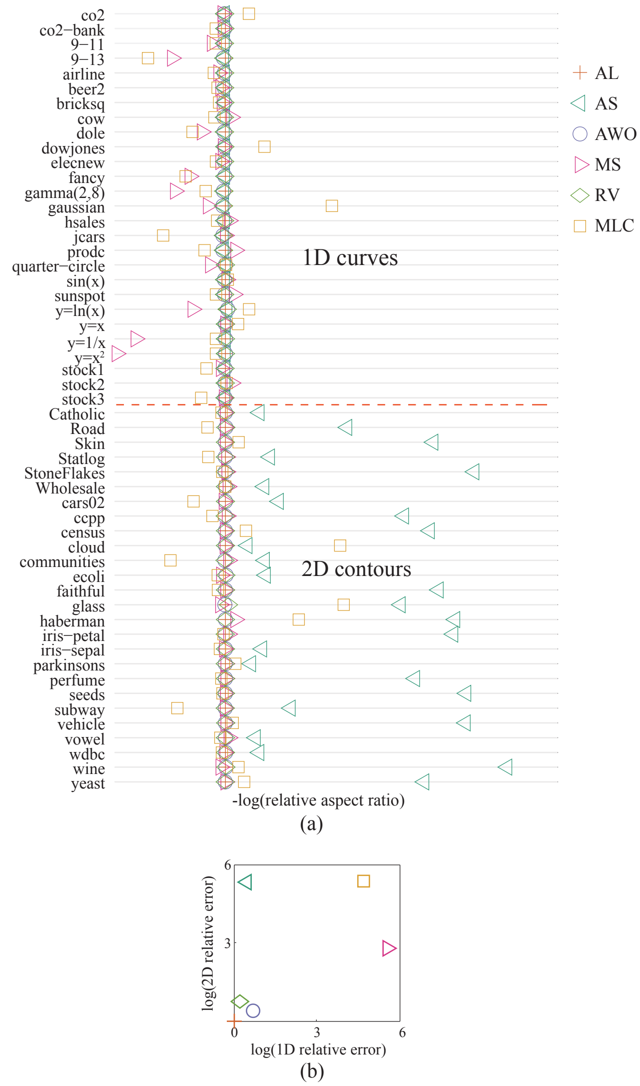

Figure 4: Comparison of aspect ratios generated by different methods. (a) The negative log relative aspect ratios for 1D curves and 2D contours; (b) The log of averaged relative errors between AL and different methods, where the x and y axes show the relative errors computed from 1D curves and 2D contours shown in (a), respectively. AWO, RV and AL are very similar, while MLC performs poorly for some data sets.

5.3 AL vs. MS

To learn whether MS behaves similarly to AL for 2D contour, we first investigate why they are similar for some contours (see Figure 4). After looking at the sorted slopes of each contour, we find almost all of them have the similar distribution as shown in Figure 9. We think the main reason is that the tested contours generated by KDE are smooth and symmetric.

To further investigate this observation, we first synthesized a circular contour which consists of 10 circles and then discretized these contours with different number of edges: 10, 20 and 30. As demonstrated in Figure 8, MS only performs similarly to AL when the number of edges is 20, while it produces larger aspect ratios in the other two cas-

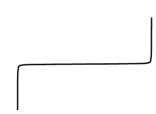

Figure 9: The slope distribution of KDE generated contours.

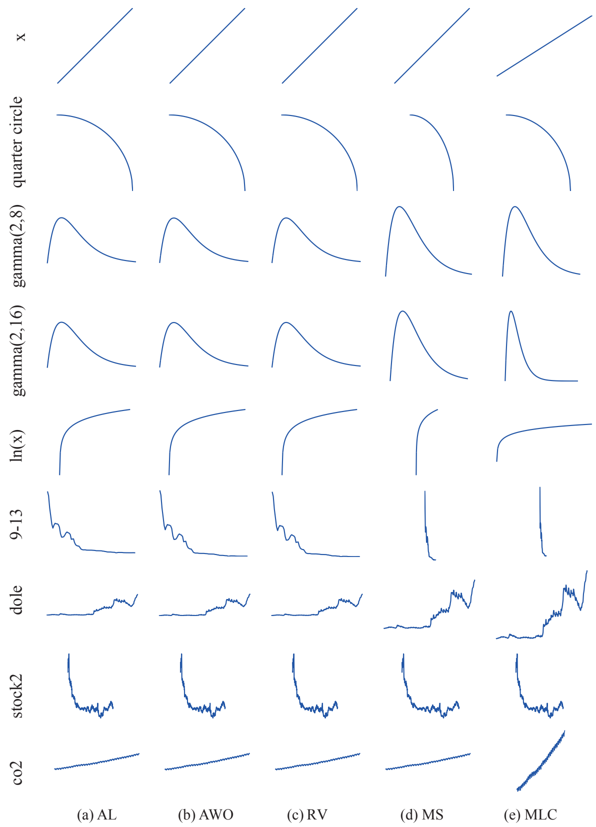

Figure 5: Selected 1D curves from Figure 4 show that AWO and RV produce almost the same aspect ratios with AL. MS selects smaller aspect ratios for most of data, while it produces overly taller aspect ratios in some cases (*ln(x)* and *9-13*). In contrast, MLC produces slightly larger aspect ratios for most of data, whereas it selects smaller ones for the *gamma(2,16)* and *9-13*.

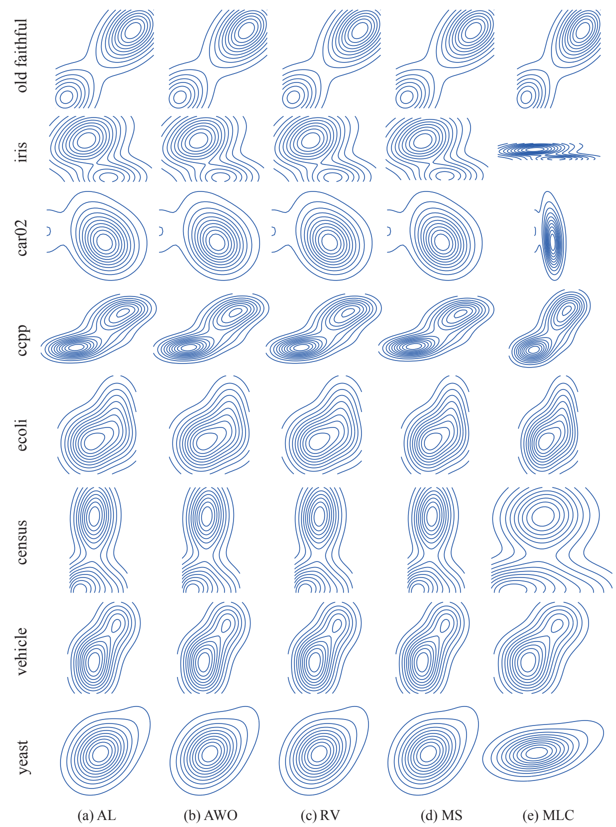

Figure 6: Selected 2D contours from Figure 4 generated by different aspect ratio selection methods. As in the case of 1D curves, AWO, RV and AL are almost the same. MS is similar to AL, but it selects slightly taller aspect ratio for *ecoli*. MLC produces smaller aspect ratios in some cases (*iris* and *census*) but selects larger ones for the other data.

es. This indicates that MS is not always similar to AL, even though Figure 8(b) and (c) have a similar distribution of the sorted slopes.

5.4 AL vs. RV

Since RV is the generalization of AS, it is inherent parameterization invariant and symmetry preservation. Because of these characteristics, the produced aspect ratios are quite close to the ones generated by AL, see Figure 2. As discussed by Talbot et al. [15], RV is faster than AL. Here we show RV is also more robust than AL.

Given a monotonically increasing curve between (0,0) and (1,1), e.g. half of a parabola, AL will not always pick an aspect ratio of 1. In contrast, RV will always pick such an aspect ratio for this curves (this can be derived directly from Equation 14). Furthermore, AL does not work in some cases, because the derivative of Equation 6 with respect to a is

$$\sum_{i}^{n} \frac{\Delta y_i^2 - \frac{1}{a^2}\Delta x_i^2}{2\sqrt{\frac{\Delta x_i^2}{a} + (a\Delta y_i)^2}},$$

and requires the denominator not to be zero. In other words, AL cannot handle data with redundant items, whose Δx and Δy are zero, while RV is robust to any kinds of inputs, unless the line is a horizontal line.

5.5 AL vs. MLC

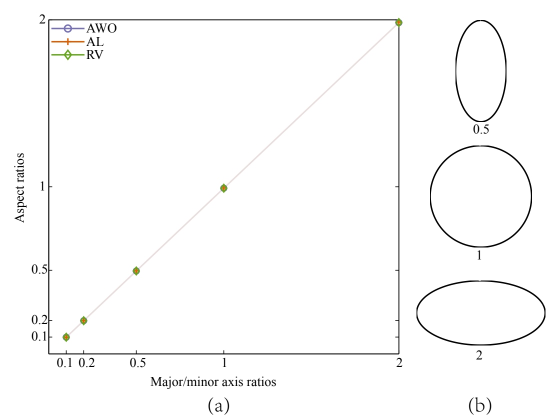

Figure 7: Comparison of AL, AWO and RV with ellipses generated by varying ratios of major to minor axis. (a) shows the relationship between the ratios of major to minor axis and the selected aspect ratios; (b)three ellipses with different ratios. The aspect ratios of all methods are almost the same, ellipses are banked to circles.

The 45° principle attempts to choose an aspect ratio so that the orientations of all line segments are centered at 45°. In some cases, however, the selected aspect ratios might result in unpleasing images. Figure 10 shows an example, where AL, AWO, and RV

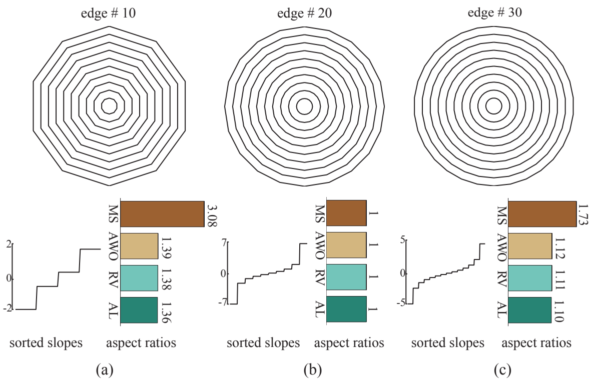

Figure 8: Comparison of the aspect ratios generated by MS, AWO, RV and AL on 2D circular contours with different discretization. MS behaves the same as AL when the number of edge is 20, while its resulted aspect ratios are taller when the number of edges are 10 and 30. In contrast, AL, AWO and RV always are very similar.

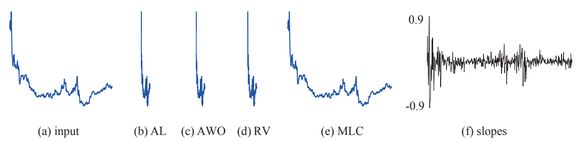

Figure 10: Banking a curve (a) where most of slope values (see (f)) are very small. AL (b), AWO (c), and RV (d) produce overly tall aspect ratios, while MLC (e) preserves the original shape.

produce overly taller aspect ratios. Instead, the aspect ratio generated by MLC preserves the original shape (see Figure 10(d)). To understand the reason for this, we investigated the difference between this curve and other tested curves and found that the slope values are quite small (see Figure 10(e)). To bank line segments to 45 degree, the aspect ratio selected by AL, AWO and RV has to be large.

However, it does not mean that the results shown in Figure 10(b,c,d) are useless. Compared to Figure 10(e), dominant features such as the one large peak or two consecutive small peaks and one valley, are more salient in Figure 10(b,c,d). This can help users to quickly see large scale trends but for investigating details they need to look at Figure 10(e). Thus, a combining these two methods can provide a proper two-scale exploration method.

6 Discussion And Future Work

In this paper, we investigated the mathematical foundations of AL. By introducing a line integral representation, we unveiled its parameterization invariance and extend previous methods to also achieve this invariance. Through establishing the mathematical connection between AL and the principle of banking to 45°, we proved the bounds of its generated average absolute slopes. Our evaluation demonstrates that AL, AWO and RV always perform very similarly. Due to the robustness and low time complexity of RV we believe that this method should become the default aspect ratio selection method.

However, the mathematical foundation of AL has not been fully understood. First, we have not strictly proven that AL really banks line segments to 45°. Second, the inequalities among the aspect ratios selected by AL, AWO and RV have not been explained yet. For most data sets they occur in the order AWO>RV>AL, while the order is AL>RV>AWO for a few algebraic curves such as $\log(x)$. Last but not the least, the relationship between AL and orientation resolution has not been explored. Whereas Talbot et al. [15] showed that AL can be derived from orientation resolution, the aspect ratios generated by AL and MLC are quite different.

The quantitative comparison demonstrates that AL, AWO and RV perform similarly, yet the perceptual reasons behind them are different. AWO is derived from the goal of banking to 45°, while RV is related to curvature-based visual perception. Nonetheless, there is no perceptual foundation for AL, although Talbot et al. [15] suggested some hypotheses. Investigating the visual cues to understand the perceptual reason of AL is a part of our ongoing work. Well-designed perceptual human evaluation will likely be necessary to learn whether the perceptual reasons behind three methods are equivalent.

Substantial future work remains to be done for fully understanding the mathematical and perceptual foundations of AL. Understanding the equivalence between banking to 45° and maximizing orientation resolution for multiple line segments may provide insights about the mathematical foundations. Extending AL and RV to select proper aspect ratios for line graphs that involve multiple time series [7] and 2D scatterplots [5] may provide more insights into their perceptual foundations.

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